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Recognition of Surfaces in Three-Dimensional Digital Images*

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This is a continuation of a series of papers on the digital geometry of three-dimensional images. In an earlier paper by Morgenthaler and Rosenfeld, a three-dimensional analog of the two-dimensional Jordan curve theorem was established. This was accomplished by defining simple surface points under the symmetric consideration of 6-connectedness and 26-connectedness and by characterizing a simple closed surface as a connected collection of "orientable" simple surface points. The necessity of the assumption of orientability, a condition of often prohibitive computational cost to establish, was the major unresolved issue of that paper. In this paper, the assumption is shown not to be necessary in the case of 6-connectedness and, unexpectedly, it is shown that the property of orientability is not symmetric with respect to the two types of connectedness.

1. INTRODUCTION

The digital geometry of three-dimensional images is a topic of considerable current interest, as a result of the increasing availability of 3-D discrete arays of data such as those produced in computed tomography. An introduction to the topological properties of 3-D images can be found in [1].

One of the interesting problems in 3-D digital topology is that of defining surfaces. Intuitively, a surface S is a set that is everywhere "thin," in the sense that in the neighborhood of any $p \in S$, there are exactly two components of $\bar{S}$ (the complement of S), and every neighbor of p in S is adjacent to both of these components. (This definition will be stated more precisely in the next section.) In [2] it was shown that surfaces defined in this way satisfy the 3-D analog of the Jordan curve theorem, i.e., they separate the space into two components, and "inside" and an "outside." However, this was shown only under the assumption that the surfaces were

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"orientable," a local property which means that for any p, the two components of $\bar{S}$ mentioned earlier remain distinct even when we enlarge the neighborhood of p. The question of whether nonorientable surfaces exist was left open in [2].

This paper shows that when we use 6-connectedness for S (see the next section), nonorientable surfaces do not exist. Furthermore, an example is given to show that surprisingly such surfaces can exist locally when we use 26-connectedness. However, in a subsequent paper by the first author, it will be shown that globally (i.e., if each point of the set satisfies the small neighborhood surface restriction) there can exist no 26-connected nonorientable surfaces.

In two dimensions, an alternative to using 4-connectedness for "objects" and 8-connectedness for "background" is to use 6-connectedness for both. This results in a pleasing symmetry, inherited from the hexagonal tessellation, which has many advantages over the square tessellation. It would be of interest to investigate the benefits of using a better tessellation of three-space, say based on dodecahedral rhomboids rather than cubes. It might also be of interest to consider other ways of defining connectedness of cubes, e.g., 18-connectedness.[1]

2. Connectivity and Simple Closed Surfaces

Let Σ denote a 3-D array of lattice points, which, without loss of generality, we may assume to be defined by integer valued triples of Cartesian coordinates (x, y, z). We consider two types of neighbors of a point $p = (x_p, y_p, z_p) \in \Sigma$:

(i) the neighbord (u, v, w) such that $|x_p - u| + |y_p - v| + |z_p - w| = 1$,

(ii) the neighbors (u, v, w) such that $\max\{|x_p - u|, |y_p - v| + |z_p - w|\} = 1$.

We refer to the neighbors of type (i) as 6-neighbor of p (the face neighbors) and to the neighbors of type (ii) as 26-neighbors of p (the face, edge, and corner neighbors). The 6-neighbors are said to be 6-adjacent to p, and the 26-neighbors are said to be 26-adjacent to p. The statement that α is a path from point p to point q in Σ means that there exists a positive integer n such that $\alpha = \{p_0, p_1, \ldots, p_n\} \subseteq \Sigma$, where $p_0 = p$, $p_n = q$ and p_i is adjacent to p_{i-1} for $1 \leqslant i \leqslant n$. The terms 6-path and 26-path are utilized depending on the type of adjacency under consideration.

[1] We are indebted to the referee for these remarks.

Let S denote a nonempty subset of Σ which, without loss of generality, we may assume does not meet the border of Σ. The points p and q of S are said to be connected in S provided there is a path from p to q which is contained in S. Connectivity is an equivalence relation, and the classes under this relation are called components. Again, the terms 6-connectivity, 26-connectivity, 6-components, and 26-components are utilized depending on the type of path under consideration.

Similarly, we can consider the components of the complement $\bar{S}$ of S. Exactly one of these components contains the border of Σ; this component is called the bachground of S. All other components of $\bar{S}$, if any, are called cavities in S. As is the custom in 2-D digital geometry, opposite types of connectivity are assumed for S and $\bar{S}$ to avoid anomalous situations.

Finally, let p be a point of S. We let $N_{27}(p)$ denote the 27 points in the $(3 \times 3 \times 3)$ neighborhood of p, and we let $N_{125}(p)$ denote the 125 points in the $(5 \times 5 \times 5)$ neighborhood centered at p. (Note that in [2], $N_{27}(p)$ and $N_{125}(p)$ were defined so as to exclude p. The change is made here in order to simplify the introduction of new notation.)

Surfaces. In [2], the above structure on the 3-D lattices was utilized to introduce the concept of a simple closed surface in order to establish a nontrivial 3-D analog of the 2-D Jordan curve theorem.

A point $p \in S$ is called a *simple surface point* provided:

(i) $S \cap N_{27}(p)$ has exactly one component adjacent to p (in the S sense); denote this component A_p.

(ii) $\bar{S} \cap N_{27}(p)$ has exactly two components, C_1 and C_2, adjacent to p (in the $\bar{S}$ sense).

(iii) If $q \in S$ and q is adjacent to p (in the S sense), then q is adjacent (in the $\bar{S}$ sense) to both C_1 and C_2.

As observed in [2], there are at most two components of $\bar{S} \cap N_{125}(p)$ adjacent (in the $\bar{S}$ sense) to a simple surface point p. Thus, suppose that p is a simple surface point of S and that each element of A_p is also a simple surface point of S (i.e., p is not near an "edge"). When $\bar{S} \cap N_{125}(p)$ has two components adjacent to p, (the surface at) p is said to be *orientable* and A_p is called a *disk*. When $\bar{S} \cap N_{125}(p)$ has only one component adjacent to p, (the surface at) p is said to be *nonorientable* and A_p is called a *cross-cap*.

THEOREM 0 [2]. *If S is a connected collection of orientable simple surface points, then S has exactly one cavity, and S is said to be a simple closed surface.*

We quote from [2], "(Theorem 0) is the 3-D analog of the Jordan curve theorem for connected sets of simple surface points. The definition given for a simple surface point is modeled after the standard definition in continuous space, namely, that a surface point is one whose neighborhood is homeomorphic with the inside of a circle on the plane. Thus every point in a small enough neighborhood of a point must be adjacent to either side of the surface.

Similarly, the concepts of orientability and cross-caps are modeled after the corresponding concepts used in the topology of continuous space. A cross-cap is homeomorphic with a Möbius strip, and may be visualized by deforming the edge of the strip to a circle in the plane. Thus, while each point on the face of the strip appears as a surface point, there is only one side (face) in the collection of points. We use the requirement on the 125-neighborhood of a surface point to guarantee that such phenomena do not occur (at least locally).

This raises the question of the realizability of cross-caps in the 3-D lattice. That is, are the definitions of connectedness, together with the definition of simple surface point, strong enough to imply that cross-caps do not exist? From a theoretical standpoint an affirmative answer to this question would simplify the definition of simple closed surface, and from a practical viewpoint it would lessen the computational cost of detecting simple closed surfaces. While various properties such as symmetries may be used to reduce the effort needed to answer this question, the answer ultimately rests on a case analysis of the 2^{124} different configurations in the 125-neighborhood of a point $p \in S$."

3. Cross-Caps in the 3-D Lattice

The purpose of this paper is to answer the quetions raised in [2] by (1) presenting in Example 1 a 26-connected cross-cap, and (2) establishing in Theorem 1 that there exist no 6-connected cross-caps in the 3-D lattice. Thus, we succeed in simplifying the definition of a 6-connected simple closed surface to that of a 6-connected collection of simple surface points.

Although Example 1 shows that the property of orientability is not ensured locally by the definition of a 26-connected simple surface point, the first author has established that this is so globally. It will be shown in a subsequent paper that a 26-connected collection S of points such that each point of S is a simple surface point is, in fact, orientable at each point. Hence, the assumption of orientability may be removed from Theorem 0 regardless of the type of connectedness under consideration.

EXAMPLE 1. A 26-connected cross-cap.

1st plane	2nd plane	3rd plane	4th plane	5th plane
0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	0 0 0 1 0	0 0 0 0 0
0 0 0 1 0	0 0 1 0 1	0 0 0 1 1	0 0 1 0 1	0 0 0 1 0
0 0 1 0 0	0 1 0 0 0	0 0 $\underline{1}$ 0 0	0 1 0 0 0	0 0 1 0 0
0 0 0 1 0	0 0 1 0 1	0 0 0 1 1	0 0 1 0 1	0 0 0 1 0
0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	0 0 0 1 0	0 0 0 0 0

It should be mentioned that the deceptively simple construction of Example 1 took longer to produce than the complex proof of Theorem 1 given below. In fact, Example 1 was derived step by step in a futile attempt to obtain a symmetric argument for 26-connectedness to that given in Theorem 1 for 6-connectedness. Perhaps the difficulty in deciding the cross-cap questions was due to the fact that the answers are so counter to the intuition established by previous results. First, one would expect the same answer for the two types of connectivity. Second, failing symmetry, one would expect that 6-connectedness, with the greater connectivity of the complement, would produce the counterexample.

Unfortunately, it is, of course, more difficult to show that something does not happen in digital geometry than to present a simple array of "0's" and "1's." The combinatorial detail involved in the following proof is unvoidable.

THEOREM 1. *There does not exist a 6-connected cross-cap.*

3.1. *Outline of the Proof*

Throughout the proof, p denotes a fixed simple surface point of the 6-connected subset S of Σ such that each point of $A_p = N_{27}(p) \cap S$ is also a simple surface point of S. Although it is not actually necessary to consider individually 2^{124} different configurations of $N_{125}(p)$ as suggested above, the reader of the following proof may very well begin to suspect that this is the case. Indeed, the fact that the realizability of cross-caps in the 3-D lattice depends on the choice of 6-connectedness or 26-connectedness witnesses the necessity for a detailed analysis beyond the symmetric definitions of simple surface points. The difficulty in deciding the issue for 6-connectedness ultimately focuses in (i) finding an overall strategy to upgrade the separation properties of $N_{27}(p)$ to include $N_{125}(p)$, and (ii) finding an efficient notation to describe the process.

3.2. *Strategy*

If M is a subset of Σ, let $\bar{M}$ denote $M \cap \bar{S}$. Furthermore, if N is a subset of Σ which contains p, let $P(N)$ denote the property that $\bar{N}$ has two 26-

components which are 26-adjacent to p. Hence, our goal is to establish inductively that $P(N)$ holds where $N = N_{125}(p)$.

(0) If $N = N_{27}(p)$, $P(N)$ holds.

(1) Lemma 1 establishes that if N is the union of the top two (3×3)-planes of $N_{27}(p)$ or the union of the bottom two (3×3)-planes of $N_{27}(p)$, then $\bar{N}$ cannot have three components adjacent to p.

(2) Lemma 2 uses Lemma 1 to establish that $P(N)$ holds, where N is the union of five (3×3)-planes centered on p.

(3) Lemma 3 uses geometric symmetry on Lemma 2 to observe that $P(N)$ holds, where N is the union of three (5×3)-planes centered on p.

(4) Lemma 4 uses Lemma 3 to establish that $P(N)$ holds, where N is the union of five (5×3)-planes centered on p.

(5) Lemma 5 uses geometric symmetry on Lemma 4 to observe that $P(N)$ holds, where N is the union of three (5×5)-planes centered on p.

(6) Lemma 6 uses Lemma 5 to establish that $P(N)$ holds, where $N = N_{125}(p) =$ the union of five (5×5)-planes centered on p.

Lemmas 2, 4, and 6 are each proved in a similar manner, but with escalating complexity, by assuming the negation and arriving at a contradiction to Lemma 1. Lemmas 2′, 4′, and 5′ are necessary technicalities.

3.3. *Notation*

For each $a = (x_a, y_a, z_a)$, let $a(i, j, k) = (x_a + i, y_a + j, z_a + k)$. In addition, let $a+$ denote $a(0, 0, 1)$ and $a-$ denote $a(0, 0, -1)$. For l, m, n odd positive integers and k an integer, let:

(1) $N^k_{m,n}(a) = \{a(i, j, k) \mid -(m-1)/2 \leqslant i \leqslant (m-1)/2$ and $-(n-1)/2 \leqslant j \leqslant (n-1)/2\}$,

(2) $N^{k=g,h}_{m,n}(a) = \bigcup_{k=g,h} N^k_{m,n}(a)$,

(3) $N^*_{m,n,l}(a) = N^{k=-(l-1)/2,(l-1)/2}_{m,n}(a)$,

(4) $N_a = N^*_{3,3,3}(a)$, $N^k_a = N^k_{3,3}(a)$, $N^{k=g,h}_a = N^{k=g,h}_{3,3}(a)$, etc.

For example,

$$
\begin{aligned}
N_a &= N^*_{3,3,3}(a) = N_{27}(a),\\
N^*_{5,5,5}(a) &= N_{125}(a),\\
N^{-2}_{5,3}(a) &= \text{the } (5 \times 3)\text{-plane centered on } a(0, 0, -2),\\
\bar{N}^{-2}_{5,3}(a) &= N^{-2}_{5,3}(a) \cap \bar{S},\\
N^*_{5,5,3}(a) &= \text{three } (5 \times 5)\text{-planes centered on } a.
\end{aligned}
$$

Finally, "#" denotes "contradiction," "w.l.o.g." denotes "without loss of generality."

3.4. *Proof*

LEMMA 1. *If a is a simple surface point of S, then two 26-components of $\bar{N}_a^0$ cannot be merged in $\bar{N}_a$. Furthermore, neither of $\bar{N}_a^{i=0,1}$ nor $\bar{N}_a^{i=-1,0}$ has three 26-components.*

Proof. Suppose two components, C_1 and C_2, of $\bar{N}_a^0$ are merged in $\bar{N}_a$. Let C_1' and C_2' denote the two components of $\bar{N}_a$, where w.l.o.g. $(C_1 \cup C_2) \subseteq C_1'$.

(1) [$C_2' \cap N_{3,3}^0(a) \neq \varnothing$, and hence $\bar{N}_a^0$ has three components.] Suppose $C_2' \cap \bar{N}_a^0 = \varnothing$. Then both $a+$ and $a-$ must be in S. To see that this is true, w.l.o.g. suppose $a+ \in \bar{S}$. Then $a-$ must be in S or else $\bar{N}_a$ would have only one compoment. Hence, since $a+$ is 6-adjacent to a and $C_2' \cap N_a^0 = \varnothing$, then $C_2' \cap N_a^{-1} \neq \varnothing$. Now, consider the two cases where (i) some element of $C_2' \cap N_a^{-1}$ is 6-adjacent to $a-$ or (ii) no element of $C_2' \cap N_a^{-1}$ is 6-adjacent to $a-$. If (i), then w.l.o.g. let $y = a(0,-1,-1) \in C_2'$. Then y cannot be 26-adjacent to an element of $\bar{S}$ which is 26-adjacent to $a+$. Thus, $\{a(-1,0,0), a(1,0,0), a(-1,-1,0), a(0,-1,0), a(1,-1,0)\} \subseteq S$. Furthermore, since now $(C_1 \cup C_2) \cap \{a(-1,1,0), a(0,1,0), a(1,1,0)\} \neq \varnothing$, it follows that $a(0,1,0) \in S$ (or else C_1 and C_2 would be 26-adjacent in N_a^0) and that both $a(-1,0,-1)$ and $a(1,0,-1)$ are in S (or else y would be 26-adjacent to $(C_1 \cup C_2) \subseteq C_1'$). However, $a(0,1,0)$ is 6-adjacent to a but it cannot now be 26-connected to $y \in C_2'$ in $\bar{N}_a$. (# to (i)) If (ii), then w.l.o.g. let $y = a(1,-1,-1) \in C_2' \cap N_a^{-1}$. Then $\{a(0,-1,0), a(1,-1,0), a(0,-1,-1), a(1,-1,-1)\} \subseteq S$. But, since C_1 and C_2 are not 26-connected in $\bar{N}_a^0$, one of $a(-1,0,0)$ and $a(0,1,0)$ must also be in S and 6-adjacent to a. Again, this is impossible for there would not be a 26-path in $\bar{N}_a$ from such a point to $y \in C_2'$. (# to (ii)) Thus $\{a+, a-\} \subseteq S$. However, since both of $a+$ and $a-$ are 6-adjacent to a, both must be 26-adjacent to C_2'. Hence, $C_2' \cap \bar{N}_{3,3}^0(a) \neq \varnothing$. (# from which (1) follows.) Note C_1, C_2, $C_2' \cap \bar{N}_a^0$ produce three components of $\bar{N}_a^0$.

(2) [The contradiction.] Since $\bar{N}_a^0$ has three components, three of $\{a(-1,0,0), a(0,1,0), a(0,-1,0), a(1,0,0)\}$ must be in S. Hence, w.l.o.g., let $\{a(-1,0,0), a(0,-1,0), a(0,1,0)\} \subseteq S$, with $y_1 = a(-1,-1,0)$ and $y_2 = a(1,-1,0)$ in different members of $\{C_1, C_2, C_2'\}$. Note that one of y_1 and y_2 is not in $(C_1 \cup C_2) \subseteq C_1'$, otherwise there could not be a 26-path in $\bar{N}_a$ connecting $a((0,-1,0)$ to C_2' without merging C_1' and C_2'. Thus, w.l.o.g. let $y_1 \in C_1$ and $y_2 \in C_2'$. Now, either (i) $a(0,1,0) \in C_2$, (ii) $a(0,1,0) \in S$ and $a(-1,1,0) \in C_2$, or (iii) $a(0,1,0) \in S$ and $a(1,1,0) \in C_2$. In either case, $a(-1,0,0)$ cannot be 26-adjacent to C_2' in N_a without merging C_1' and

C_2'. Thus, two 26-components of N_a^0 cannot be merged in $\bar{N}_a$. Since $\bar{N}_a$ has two components, it follows immediately that neither of $\bar{N}_a^{i=0,1}$ or $\bar{N}_a^{i=-1,0}$ can have three components. The proof is complete.

LEMMA 2. *$\bar{N}^*_{3,3,5}(p)$ has two components each of which is 26-adjacent to p.*

Proof. (We do more work than necessary here to establish the format for the proofs of Lemmas 4 and 6.) (1) [$\bar{M}$ has two components which are 26-adjacent to p, where $M = N_p \cup N_p^2$.] Suppose not. Since $\bar{N}_p$ has two components, C_1 and C_2, which are 26-adjacent to p, there must exist a 26-path α in $\bar{N}_p^2$ from $y_1 \in N_p^1 \cap C_1$ to $y_2 \in N_p^1 \cap C_2$.

(i) [$p+ \in S$]. Otherwise, $\bar{N}_p^1$ could have only one component.

(ii) [$\bar{N}_{p+}$ has two components, C_1' and C_2', which are 26-adjacent to p.] Since $p+ \in A_p$, $p+$ is also a simple surface point of S. Hence, $\bar{N}_{p+}$ has two components 26-adjacent to $p+$. But p is 6-adjacent to $p+$ in N_{p+}; thus both these components must be 26-adjacent to p.

(iii) [Each of y_1 and y_2 is 26-connected to p in $\bar{N}_p^{i=0,1}$.] Obvious.

(iv) [The contradiction from which (1) follows.] Since y_1 and y_2 are 26-connected by $\alpha \subseteq \bar{N}_p^2$, $\{y_1, y_2\}$ is contained in C_1' or C_2', say C_1'. However, now $C_1 \cap N_p^{i=0,1}$, $C_2 \cap N_p^{i=0,1}$, and $C_2' \cap N_p^{i=0,1}$ produce three components of $\bar{N}_p^{i=0,1}$ in contradiction to Lemma 1.

(2) [$\bar{N}^*_{3,3,5} = \bar{M} \cup N_p^{-2}$ has two components which are 26-adjacent to p.] This statement follows immediately from a symmetric argument to that given in (1).

LEMMA 2′. *Suppose a is a simple surface point of S and each of $a+$ and $a-$ is either in $\bar{S}$ or is also a simple surface point of S; then $\bar{N}^*_{3,3,5}(a)$ has two components which are 26-adjacent to a.*

Proof. This follows immediately from the proof of Lemma 2.

LEMMA 3. *$\bar{N}^*_{5,3,3}(p)$ has two components which are 26-adjacent to p.*

Proof. Geometric symmetry to Lemma 2.

LEMMA 3′. *Consider N_a, where a is a simple surface point of S, and $b = a(-1, 0, 0)$, $b_1 = a(-1, 0, 1)$, $b_2 = a(0, 0, -1)$ are all in S, and $D \cap C_1 \neq \varnothing$, where C_1 and C_2 are the two components of $\bar{N}_a$ and $D = \{a(1, 1, 0), a(1, 0, 0), a(1, -1, 0)\}$. Then each of b and b_1 is 26-adjacent to $C_1 \cap M$, where $M = N_a^{i=-1,0}$.*

Proof. Suppose N_a and M are as above and b is not 26-adjacent to $C_1 \cap M$. (Observe that b_1 is 26-adjacent to $C_1 \cap M$ if and only if b is 26-adjacent to $C_1 \cap M$.) Let $H = \{a(-1,1,0),\ a(0,1,0),\ a(1,1,0),\ a(-1,1,-1),\ a(0,1,-1),\ a(1,1,-1)\}$ and $K = \{a(-1,-1,0),\ a(0,-1,0),\ a(1,-1,0),\ a(-1,-1,-1),\ a(0,-1,-1),\ a(1,-1,-1)\}$. Since $\{a, b, b_1, b_2\} \subseteq S$ and $D \cap C_1 \neq \varnothing$, observe that if $H \cap C_2 \neq \varnothing$ and $K \cap C_2 \neq \varnothing$, there is no 26-path in $M \cap C_2$ from $H \cap C_2$ to $K \cap C_2$.

(1) [Claim: $H \cap C_2 \neq \varnothing$ and $K \cap C_2 \neq \varnothing$.] Suppose $H \cap C_2 = \varnothing$. By assumption b is not 26-adjacent to $C_1 \cap M$. But since b is 6-adjacent to a, there must exist a path α in $N_a^1 \cap C_1$ from b to $D \cap C_1$. Also, since $H \cap C_2 = \varnothing$, $x = a(0,1,0) \in S$. Hence, x must be 26-adjacent to C_2. But since $D \cap C_1 \neq \varnothing$, $C_2 \cap \{a(1,0,0), a(1,0,-1), a(1,0,1)\} = \varnothing$ and it follows that x must be 26-adjacent to some $x_1 \in N_a^1 \cap C_2$. Now, since $C_1 \cap N_a^1 \neq \varnothing$ and $C_2 \cap N_a^1 \neq \varnothing$, $a(0,0,1) \in S$. Therefore, $a(0,-1,1) \in \alpha$; this is the only remaining possibility for a 26-path from b to $D \cap C_1$. Hence, $x_2 = a(0,-1,0) \notin C_2$ since it is 26-adjacent to α, and $x_2 \notin C_1$ since b is not 26-adjacent to $C_1 \cap M$. Thus, $x_2 \in S$, and being 6-adjacent to a, x_2 must be 26-adjacent to C_2. Since $a(0,-1,1) \in C_1$, x_2 must be 26-adjacent to some point x_3 in $N_a^{-1} \cap K \cap C_2$. However, now there is no possible path in $N_a \cap C_2$ from x_1 to x_3. # Hence $H \cap C_2 \neq \varnothing$, and similarly $K \cap C_2 \neq \varnothing$.

(2) [The contradiction.] From the above, it follows that $M \cap C_1$, $H \cap C_2$, and $K \cap C_2$ produce at least three components of $\bar{N}_a^{i=-1,0}$ in contradiction to Lemma 1.

LEMMA 4. *$\bar{N}^*_{5,3,5}(p)$ has two components which are 26-adjacent to p.*

Proof. (1) [$\bar{M}$ has two components which are 26-adjacent to p, where $M = N^*_{5,3,3}(p) \cup N^2_{5,3}(p)$.] Suppose not. From Lemma 3, $\bar{N}^*_{5,3,3}(p)$ has 2 components, C_1 and C_2, which are 26-adjacent to p. Thus if $\bar{M}$ has only one component 26-adjacent to p, there must exist a 26-path α in $\bar{N}^2_{5,3}(p)$ from $y_1 \in C_1 \cap N^1_{5,3}(p)$ to $y_2 \in C_2 \cap N^1_{5,3}(p)$.

(i) [$p+ \in S$.] Suppose $p+ \in \bar{S}$. Let $a_1 = p(-1,0,1)$ and $a_2 = p(1,0,1)$. Now $p+$ must be in one of C_1 and C_2, say $p+ \in C_1$. Then either (a) $\{y_1, y_2\}$ is contained in the leftmost column of $N^1_{5,3}(p)$, (b) $\{y_1, y_2\}$ is contained in the rightmost column of $N^1_{5,3}(p)$, or (c) α is 26-adjacent to $p+$ and we can assume $p+ = y_1$. In either case, it follows that $\{y_1, y_2\} \subseteq \bar{N}_a^0$ and y_1 is 26-connected to y_2 in $\bar{N}_a^1$, where $a = a_2$ or $a = a_2$. However, since y_1 and y_2 are not 26-connected in $\bar{M}$, $a \in N_p \cap S$, and hence a is a simple surface point of S. But $\bar{N}_a^{i=-1,0} \subseteq \bar{M}$, and the above situation violates Lemma 1. (# from which (i) follows).

(ii) [$\bar{N}^*_{5,3,3}(p+)$ has two components, C_1' and C_2', which are 26-adjacent to p.] Since $p+$ is a simple surface point of S and each of

$p+(-1,0,0)$ and $p+(1,0,0)$ is either in $\bar{S}$ or is a simple surface point of S, it follows from geometric symmetry to Lemma 2′ that $\bar{N}^*_{5,3,3}(p+)$ has two such components which are 26-adjacent to $p+$. Furthermore, since p is 6-adjacent to $p+$ in N_{p+}, each of these two components must be 26-adjacent to p.

(iii) [Each of y_1 and y_2 is 26-connected to p in $\bar{N}^{i=0,1}_{5,3}(p)$.] Consider y_1 arbitrarily. If $y_1 \in N^0_{p+}$, we are finished. Hence, w.l.o.g. let y_1 be in the leftmost column of $N^1_{5,3}(p)$. If y_1 is not 26-connected to p in $\bar{M}$, then $p(1,0,1)$ and $p(1,0,0)$ must be in S. However, if we assign $a=p(1,0,1)$, $b=p+$, $b_1=p$, and $b_2=p(1,0,0)$ and then apply Lemma 3′ with $y_1 \in D \cap C_1$, it follows that y_1 is 26-connected to p in $\bar{N}^{i=-1,0}_{p(1,0,1)} \subseteq \bar{N}^{i=0,1}_{5,3}(p)$. (# from which (iii) follows.)

(iv) [The contradiction from which (1) follows.] Since y_1 and y_2 are 26-connected in $\bar{N}^*_{5,3,3}(p+)$, w.l.o.g. let $\{y_1, y_2\} \subseteq C'_1$. Now, p must be 26-adjacent to $C'_2 \cap \bar{N}^{i=0,1}_p$. However, $C_1 \cap \bar{N}^{i=0,1}_p$, $C_2 \cap \bar{N}^{i=0,1}_p$, and $C'_2 \cap \bar{N}^{i=0,1}_p$ produce at least three components of $\bar{N}^{i=0,1}_p$ in contradiction to Lemma 1.

(2) It now follows immediately by a symmetric argument to that given above that $\bar{N}^*_{3,3,5}(p) = \bar{M} \cup \bar{N}^{-2}_{5,3}(p)$ has two components which are 26-adjacent to p.

LEMMA 4′. *Suppose a is a simple surface of S and each of $\{a(-1,0,i), a(0,0,i), a(1,0,i) \mid i=0,3\}$ is either in $\bar{S}$ or is also a simple surface point of S; then $\bar{N}^*_{5,3,5}(a+)$ has two components which are 26-adjacent to a.*

Proof. Suppose not. Note that $\bar{N}^*_{5,3,3}(a)$ has two components 26-adjacent to a by geometric symmetry to Lemma 2′, and then so does $\bar{N}^{i=-1,2}_{5,3}(a) = \bar{N}^*_{5,3,3}(a) \cup \bar{N}^0_{5,3}(a(0,0,2))$ by the proof of Lemma 4. Furthermore, if $a+ \in S$ or $a(0,0,2) \in \bar{S}$, then the proof is finished as in the proof of Lemma 4. Thus, suppose $a+ \in \bar{S}$, $a(0,0,2) \in S$, and there is a 26-path α in $\bar{N}^1_{5,3}(a(0,0,2))$ from $y_1 \in C_1 \cap \bar{N}^0_{5,3}(a(0,0,2))$ to $y_2 \in C_2 \cap \bar{N}^0_{5,3}(a(0,0,2))$, where C_1 and C_2 are the two components of $\bar{N}^{i=-1,2}_{5,3}(a)$ 26-adjacent to a. w.l.o.g. let $a+ \in C_1$. Then y_2 must be in either the rightmost or leftmost column of $N^0_{5,3}(a(0,0,2))$, say rightmost. Furthermore, $\{a(1,0,2), a(1,0,1)\} \subseteq S$, and by Lemma 1, y_2 and $a+$ are in opposite components of $\bar{N}_{a(1,0,2)}$. However, $a(0,0,2)$ is 6-adjacent to $a(1,0,2)$ and, by geometric symmetry to Lemma 2′, $\bar{N}^*_{5,3,3}(a(0,0,2))$ has two components 26-adjacent to $a(0,0,2)$. Hence, $y_1 \notin \bar{N}^0_{a(1,0,2)}$ and y_2 and $a+$ are also in opposite components of $\bar{N}^*_{5,3,3}(a(0,0,2))$. Therefore y_1 and $a+$ are in opposite components of $\bar{N}^*_{5,3,3}(a(0,0,2))$, y_1 is in the leftmost column of $N^0_{5,3}(a(0,0,2))$, $\{a(-1,0,1), a(-1,0,2)\} \subseteq S$, and y_1 and $a+$ are in opposite components of both $\bar{N}_{a(-1,0,2)}$ and $\bar{N}_{a(-1,0,1)}$. Now, since there is a path in $\bar{N}^{i=-1,2}_{5,5,3}(a)$ from

a to y_1, we have y_1 26-connected to y_1' in $\bar{N}^{i=-1,0}_{a(-1,0,1)}$. Thus, $a(-1,0,0) \in S$, and y_1' and $a+$ are in opposite components of $\bar{N}_{a(-1,0,0)}$. However, it now follows by geometric symmetry to Lemma 2′ that y_1' and $a+$ are in opposite components of $\bar{N}^*_{5,3,3}(a)$ and thus of $\bar{N}^{i=-1,2}_{5,3}(a)$. Since $\{a+, y_1'\} \subseteq C_1$, this is a contradiction from which the lemma follows. Hence $\bar{N}^{i=-1,3}_{5,3}(a) = \bar{N}^*_{5,3,5}(a+)$ has two components 26-adjacent to a.

Lemma 5. *$\bar{N}^*_{5,5,3}(p)$ has two components 26-adjacent to p.*

Proof. Geometric symmetry to Lemma 4.

Lemma 5′. *Suppose a and b are simple surface points of S, $\bar{N}^*_{5,5,3}(a)$ has two components, C_1 and C_2, 26-adjacent to a, and $b \in N^0_a$ is 6-connected to a via simple surface points in N^0_a; then $C_1' \subseteq C_1$ and $C_2' \subseteq C_2$, where C_1' and C_2' are the two components of $\bar{N}_b$ 26-adjacent to b.*

Proof. Follows immediately from the definition of simple surface point.

Lemma 6. *$\bar{N}^*_{5,5,5}(p)$ has two components which are 26-adjacent to p.*

Proof. (1) [$\bar{M}$ has two components which are 26-adjacent to p, where $M = N^*_{5,5,3}(p) \cup N^2_{5,5}(p)$.] Suppose not. $\bar{N}^*_{5,5,3}(p)$ has two components, C_1 and C_2, 26-adjacent to p by Lemma 5. Hence, if $\bar{M}$ has only one component 26-adjacent to p, there must exist a path α in $\bar{N}^2_{5,5}(p)$ from $y_1 \in C_1 \cap N^1_{5,5}(p)$ to $y_2 \in C_2 \cap N^1_{5,5}(p)$.

(i) [$p+ \in S$.] Suppose $p+ \in \bar{S}$. Then one of y_1 and y_2 cannot be 26-connected to p in $\bar{N}^{i=0,1}_{5,5}(p)$ or else y_1 would be 26-connected to y_2 in $\bar{N}^*_{5,5,3}(p)$ via $p+$. Furthermore, one of y_1 an y_2 must be 26-connected to p in $\bar{N}^{i=0,1}_{5,5}(p)$. If not, there would exist $\{q_1, q_2\} \subseteq A_p \cap N^0_p$ such that each of q_1 and q_2 is 6-connected to p in $N^0_p \cap S$, y_1 and $p+$ are in opposite components of $\bar{N}_{q_1}$, and y_2 and $p+$ are in opposite components of $\bar{N}_{q_2}$. But since $\bar{N}_{q_1} \cup \bar{N}_{q_2} \subseteq \bar{N}^*_{5,5,3}(p)$, this contradicts Lemma 5′. (To see that there exist such points q_1 and q_2, suppose that y_1 is not 26-connected to p in $\bar{N}^{i=0,1}_{5,5}(p)$. If y_1 is not contained in $Z = \{p(-2,-2,1), p(-2,2,1), p(2,-2,1), p(2,2,1)\}$, then there must exist $q_1 \in N^0_p \cap S$ which is 6-adjacent to p such that $y_1 \in \bar{N}^1_{q_1}$, or else y_1 would be 26-connected to $p+$ in $\bar{N}^{i=0,1}_{5,5}(p)$. Also, since y_1 and $p+$ are in different components of $\bar{N}^{i-0,1}_{q_1}$ by Lemma 1, they are in opposite components of $\bar{N}_{q_1}$. If y_1 is contained in Z, then w.l.o.g. let $y_1 = p(2,2,1)$. Again, $q_1 = p(1,1,0)$ and $x = p(1,1,1)$ must be in S, or else y_1 would be 26-connected to $p+$ in $N^{i=0,1}_{5,5}$. Now, if both of $p(0,1,0)$ and $p(1,0,0)$ were in $\bar{S}$, p would be in $N_x \cap S$ but not 6-connected to x in A_x. Hence, since x is a simple surface point of S, it follows that one of $p(0,1,0)$ and $p(1,0,0)$ must be in S. Thus, q_1 is 6-connected to p in $N^0_p \cap S$, and, as before, y_1 and $p+$ are in opposite components of $\bar{N}_{q_i}$. The existence of q_2 is established in a similar manner, if one suppose y_2 is not 26-connected to p in $\bar{N}_{5,5}^{i=0,1}(p)$.) Hence, w.l.o.g. assume y_1 is not 26-connected to p in $\bar{N}_{5,5}^{i=0,1}(p)$ but that y_2 is so connected. Now, consider y_1. As above, if (a) y_1 is 26-connected to $y_1' \in \bar{N}_{5,5}^1(p)\backslash Z$ in $\bar{N}_{5,5}^1(p)$, then there must exist $q \in A_p \cap N_p^1$ such that q is 6-adjacent to $p+$ and y_1' and $p+$ are in opposite components of $\bar{N}_q$. If (b) y_1 is not 26-connected to $\bar{N}_{5,5}^1(p)\backslash Z$ in $\bar{N}_{5,5}^1(p)$, then w.l.o.g. let $y_1 = p(2, 2, 1)$ and note that $\{p(1, 1, 1), p(1, 2, 1), p(2, 0, 1)\} \subseteq S$. Also, y_1 and $p+$ are in opposite components of $\bar{N}_{p(1,1,1)}$. Since α must connect y_1 and y_2 in $\bar{N}_{5,5}^2(p)$, there exists $y_1' \in \{p(1, 2, 2), p(2, 0, 2)\} \cap \bar{S}$. w.l.o.g. let $y_1' = p(2, 0, 2)$. Then it follows that $q = p(1, 0, 1) \in S$ or else y_1 and $p+$ would be 26-connected in $\bar{N}_{p(1,1,1)}$. Furthermore, since q is 6-adjacent to $p(1, 1, 1)$, y_1' and $p+$ are in oppisite components of $\bar{N}_q$ by geometric symmetry to Lemma 1. Hence, either in case (a) or (b) we have that there exists $q \in A_p \cap N_p^1$ such that q is 6-adjacent to $p+$ and $p+$ and y_1' are in opposite components of $\bar{N}_q$, where y_1' is 26-connected to y_1 in $\bar{N}_{5,5}^{i=1,2}(p)$. However, by geometric symmetry to Lemma 4′, it then follows that $\bar{N}_{5,5,5}^*(p+)$ must have two components 6-adjacent to q. Thus, y_1 and $p+$ (hence y_1 and y_2) are not 26-connected by a path in $\bar{N}_{5,5}^2(p)$. (# from which (i) follows.)

(ii) [Each of y_1 and y_2 is 26-connected to p in $\bar{N}_{5,5}^{i=0,1}(p)$.] Suppose arbitrarily that y_1 is not so connected to p. From geometric symmetry to Lemma 4′, there exist two components, C_1' and C_2' of $\bar{N}_{5,5,3}^*(p+)$ 26-adjacent to $p+$. Then from Lemma 3′ as applied in the proof of Lemma 4, $y_1 \notin \{p(i,j, 1) \mid i = \{-1, 0, 1\}, j \in \{2, 2\}\} \cup \{p(i,j, 1) \mid i \in \{-2, 2\}, j \in \{-1, 0, 1\}\}$. Thus w.l.o.g. let $y_1 = p(2, 2, 1)$ and observe that $q = p(1, 1, 0)$ and $q+ = p(1, 1, 1)$ must be in S (hence, also in A_p) or else y_1 would be 26-connected to p in $\bar{N}_{5,5}^{i=0,1}(p)$. Also, $x_1 = p(1, 2, 1)$ and $x_2 = p(2, 1, 1)$ must be in S or y_1 would again be 26-connected to p in $\bar{N}_{5,5}^{i=0,1}(p)$ by application of Lemma 3′. Now consider $H = \{p(0, 1, 1), p(1, 0, 1), p(0, 1, 0), p(1, 0, 0)\}$. Let A and B denote the two components of $\bar{N}_{q+}$ 26-adjacent to $q+$, where $y_1 \in A$. $[H \cap A = \varnothing.]$ If $H \cap A \neq \varnothing$, a 26-path in $\bar{N}_{q+}^1$ would merge two components of $\bar{N}_{q+}^{i=-1,0}$ in contradiction to Lemma 1. $[H \cap B \neq \varnothing.]$ Suppose not. Then $H \subseteq S$. If $x_3 = p(1, 2, 0) \in \bar{S}$, then $x_3 \in A$ and one of $K = \{p(0, 1, 2), p(0, 2, 2)\}$ must be in B since x_1 is in S and is 6-adjacent to $q+$. Thus $y = p(2, 1, 2) \in B$ since there is a 26-path from y_1 to y_2 in $\bar{N}_{5,5,5}^2(p)$. But q must be 26-adjacent to B and $x = p(2, 0, 0)$ is the only remaining possibility which implies that $p(2, 0, 1) \in S$. However, observe that now there is no path from x to K in $\bar{N}_q$ which is impossible since $K \cap B \neq \varnothing$. Hence $x_3 \in S$. Now, by an application of Lemma 3′, it follows that y_1 is 26-connected to $p(0, 1, 0)$ by a path in $\bar{N}_{q+}^{i=-1,0}$, which is impossible since $(H \cup \{x_3\}) \subseteq S$. Hence, $H \cap B \neq \varnothing$. Let $t_2 \in H \cap B$. Now consider N_{q+}. One of $p(0, 1, 1)$ and $p(1, 0, 1)$ must be in S, or else $\bar{N}^1_{q+}$ would merge two components of $\bar{N}^{i=-1,0}_{q+}$. Hence, since $q+$ is 6-connected to $p+$ in $N^0_{p+} \cap S$, by Lemma 5′, w.l.o.g. $A \subseteq C_1'$ and $B \subseteq C_2'$. Note $t_2 \in C_2'$, and since $\bar{N}^{-1}_q$ cannot merge two components of $\bar{N}^{i=0,1}_q$, $t_2 \in C_2$.

(a) Suppose y_2 is not 26-connected to p in $\bar{N}^{i=0,1}_{5,5}(p)$. Then, as above, there exists $t_1 \in N^{i=-1,0}_{p+} \cap C_1$ such that t_1 is in one of C_1' and C_2 and y_2 is in the other. Thus, since y_1 is 26-connected to y_2 in $\bar{N}^*_{5,5,3}(p+)$, $\{y_1, y_2\} \subseteq C_1'$ and $\{t_1, t_2\} \subseteq C_2'$. But $\{t_1, t_2\} \subseteq N^{i=-1,0}_{p+}$, and t_1 is not 26-connected to t_2 in $\bar{N}^{i=-1,0}_{p+}$. Hence, by Lemma 1, t_1 is not 26-connected to t_2 in $N^*_{5,5,3}(p+)$. (# to (a)).

(b) Suppose y_2 is 26-connected to p in $\bar{N}^{i=0,1}_{5,5}(p)$. Let $t_1 \in N^{i=0,1}_p \cap C_2$ such that t_1 is 26-connected to y_2 in $\bar{N}^{i=0,1}_{5,5}(p)$. Since $p+ \in \bar{S}$, there exists $t_3 \in N^{i=0,1}_p \cap C_1$. Consider $\{t_1, t_2, t_3\} \subseteq N^{i=0,1}_p$. Note $\{t_1, t_2\} \subseteq C_2$ and therefore neither is 26-adjacent to $t_3 \in C_1$ in $\bar{N}^{i=0,1}_p$. Also, t_2 is not 26-connected to y_1 (hence to y_2) in $N^*_{5,5,3}(p+)$. Thus t_2 is not 26-connected to t_1 in $\bar{N}^{i=0,1}_p$. Hence, there are three components of $\bar{N}^{i=0,1}_p$ in contradiction to Lemma 1. (# from which (ii) follows.)

(iii) [$\bar{N}^*_{5,5,3}(p+)$ has two components, C_1' and C_2', 26-adjacent to p.] As noted above, this follows immediately from geometric symmetry to Lemma 4′.

(iv) [The contradiction to from which (1) follows.] Since y_1 and y_2 are 26-connected in $\bar{N}^*_{5,5,3}(p+)$, w.l.o.g. let $\{y_1, y_2\} \subseteq C_1'$. Now, p must be 26-adjacent to $C_2' \cap \bar{N}^{i=0,1}_p$. However, $C_1 \cap \bar{N}^{i=0,1}_p$, $C_2 \cap \bar{N}^{i=0,1}_p$, and $C_2' \cap \bar{N}^{i=0,1}_p$ produce at least three components of $\bar{N}^{i=0,1}_p$ in contradiction to Lemma 1.

(2) It now follows immediately by a symmetric argument to that given above that $\bar{N}^*_{5,5,5}(p) = \bar{M} \cup \bar{N}^{-2}_{5,5}(p)$ has two components which are 26-adjacent to p.

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